

A tight Erdős-Pósa function for long cycles

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Abstract

A classic result of Erdős and Pósa says that any graph contains either k vertex-disjoint cycles or can be made acyclic by deleting at most $O(k \log k)$ vertices. Here we generalize this result by showing that for all numbers k and l and for every graph G , either G contains k vertex-disjoint cycles of length at least l , or there exists a set X of $O(kl + k \log k)$ vertices that meets all cycles of length at least l in G . As a corollary, the tree-width of any graph G that does not contain k vertex-disjoint cycles of length at least l is of order $O(kl + k \log k)$. These results improve on the work of Birmelé, Bondy and Reed '07 and Fiorini and Herinckx '14 and are optimal up to constant factors.

1 Introduction

Let \mathcal{F} be any family of graphs. Given a graph G , a subset $X \subseteq V(G)$ is called a *transversal* (of \mathcal{F}) if the graph $G - X$ obtained by deleting X does not contain any member of \mathcal{F} . We say that \mathcal{F} has the *Erdős-Pósa property* if there exists a function $f: \mathbb{N} \rightarrow \mathbb{R}$ such that every graph G which does not contain k vertex-disjoint members of \mathcal{F} contains a transversal of size at most $f(k)$.

The study of this property dates back to 1965 when Erdős and Pósa [9] showed the following:

Theorem. *Every graph contains either k vertex-disjoint cycles or a set of at most $f(k) = (4 + o(1))k \log k$ vertices meeting all its cycles.*

The value of $f(k)$ in this theorem is optimal up to the constant factors. The Erdős-Pósa property is closely related to classical ‘covering vs. packing’ results in graph theory, such as König’s theorem or Menger’s theorem. For example, König’s theorem can be stated as follows: every bipartite graph contains either k vertex-disjoint edges or a set of $f(k) = k$ vertices meeting all the edges. The above result has spawned a long line of papers about the duality between packing and covering of different families of graphs, directed graphs, hypergraphs, rooted graphs, and other combinatorial objects (see a recent survey of Raymond and Thilikos [17] for more information).

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In this paper, we are interested in the Erdős-Pósa property for the family $\mathcal{F}_l = \{C_m \mid m \geq l\}$ of cycles of length at least l . A 1988 result of Thomassen [18] implies that for every l , the family \mathcal{F}_l has the Erdős-Pósa property with a function $f(l, k) \in 2^{l^{\mathcal{O}(k)}}$ (though recent results of Chekuri and Chuzhoy make it possible to substantially improve the dependency on k in this bound [6]). This result was sharpened by Birmelé, Bondy, and Reed [3] to $f(l, k) \in \mathcal{O}(lk^2)$ in 2007 and by Fiorini and Herinckx [11] to $f(l, k) \in \mathcal{O}(lk \log k)$ in 2014. In this paper we improve these results to the asymptotically optimal bound $f(l, k) \in \mathcal{O}(kl + k \log k)$, thus settling the question asked in [3] and [11].

Theorem 1.1. *For every integer $l \geq 3$, the family \mathcal{F}_l of cycles of length at least l has the Erdős-Pósa property with the function*

$$f(l, k) = \begin{cases} 6kl + 10k \log_2 k + 40k + 10k \log_2 \log_2 k & \text{if } k \geq 2, \\ 0 & \text{if } k = 1. \end{cases}$$

Certainly the constant factors in Theorem 1.1 are not optimal. Birmelé, Bondy, and Reed [3] conjectured the correct function in the case $k = 2$ to be $f(l, 2) = l$. The complete graph on $2l - 1$ vertices shows that, if true, this bound would be tight. Lovász [13] confirmed the conjecture for $l = 3$, while Birmelé [1] confirmed the cases $l = 4$ and $l = 5$. For larger l , Birmelé, Bondy, and Reed [3] proved that the optimal function satisfies $f(l, 2) \leq 2l + 3$. This was recently improved by Meierling, Rautenbach and Sasse [15] to $f(l, 2) \leq 5l/3 + 29/2$.

There are two constructions which together imply that the function in Theorem 1.1 is asymptotically optimal for large k and l . On the one hand, for all k and l we must have $f(l, k) \geq (k - 1)l$, as can be seen from the example of a complete graph on $kl - 1$ vertices: this graph does not contain k vertex-disjoint cycles of length at least l , but to remove all cycles of length at least l one must delete $kl - 1 - (l - 1) = (k - 1)l$ vertices. This construction also gives the lower bound $f(l, k) \geq \frac{1}{2}(k - 1) \log_2 k$ whenever $l \geq \frac{1}{2} \log_2 k$.

On the other hand, for $l < \frac{1}{2} \log_2 k$ we can obtain the lower bound $f(l, k) \geq \frac{1}{8}k \log_2 k$ using the fact that there exist 3-regular graphs on n vertices with girth at least $(1 - o(1)) \log_2 n$ [10]. Indeed for n large enough, let G denote such a graph with girth $g(G)$. Clearly G contains at most $n/g(G)$ vertex disjoint cycles. So fix $k = \lfloor n/g(G) \rfloor + 1 > n/g(G)$ and observe that for n large enough $n \geq \frac{1}{2}k \log_2 k$. All cycles in G have length at least $g(n) > \frac{1}{2} \log_2 k > l$. Thus if X is a transversal of all cycles of length at least l then $G - X$ is a forest. Because G is 3-regular, removing $|X|$ vertices leaves at least $3n/2 - 3|X|$ edges. Since the resulting graph should be a forest, we need $\frac{3}{2}n - 3|X| \leq n - |X|$, and therefore every transversal must have size $|X| \geq \frac{n}{4} \geq \frac{1}{8}k \log_2 k$. This gives the desired lower bound $f(l, k) \geq \frac{1}{8}k \log_2 k$.

Notation All graphs are assumed to be simple unless stated otherwise. However, multigraphs do make an appearance in the proof. We define a multigraph M in the standard way, that is, as an ordered pair (V, E) , where V denotes the vertex set of M and E is the multiset of edges of M . For a (multi-)graph G we denote by $V(G)$ and $E(G)$ the vertex set and the edge (multi-)set of G , respectively. Given two multigraphs M_1 and M_2 we write $M_1 \cup M_2$ for the multigraph $M = (V, E)$ where $V = V(M_1) \cup V(M_2)$ and $E = E(M_1) \cup E(M_2)$. In particular, the multiplicity of an edge e in $M_1 \cup M_2$ is

equal to the sum of multiplicities of e in M_1 and M_2 . We use the standard asymptotic notation \mathcal{O} , o , ω and Ω .

Tree-width Our results also imply an asymptotically optimal upper bound on the tree-width $\text{tw}(G)$ of every graph G that does not contain k vertex-disjoint cycles of length at least l . We need the following theorem.

Theorem 1.2 (Birmelé [2]). *Suppose that G does not contain a cycle of length at least l . Then $\text{tw}(G) \leq l - 2$.*

Generalizing this, Birmelé, Bondy, and Reed proved that any graph G not containing k vertex-disjoint cycles of length at least l has tree-width in $\mathcal{O}(k^2 l)$ [3]. Theorem 1.1 allows us to improve this bound:

Corollary 1.3. *Assume that G does not contain k vertex-disjoint cycles of length at least l . Then $\text{tw}(G) \in \mathcal{O}(kl + k \log k)$.*

Proof. Assume that G does not contain k vertex-disjoint cycles of length at least l . By Theorem 1.1 there is a set $X \subseteq V(G)$ of size $|X| \leq 6kl + 10k \log_2 k + 40k + 10 \log_2 \log_2 k$ such that $G - X$ does not contain a cycle of length at least l . By Theorem 1.2 we have $\text{tw}(G - X) \leq l - 2$. We can turn a tree-decomposition of $G - X$ into a tree-decomposition of G by adding X to each bag, which gives the bound $\text{tw}(G) \leq \text{tw}(G - X) + |X| \leq (6k + 1)l + 10k \log_2 k + 40k + 10 \log_2 \log_2 k - 2$. \square

This is tight in the sense that there are examples of graphs that do not contain k disjoint cycles of length at least l and whose tree-widths are in $\Omega(kl + k \log k)$.

In fact, similar constructions as above work. An example where $\text{tw}(G) \geq kl - 2$ is provided by the complete graph on $kl - 1$ vertices. For $l \leq c \log k$, for sufficiently small positive constant c , we can use fact that there exist constant-degree expander graphs G on n vertices with $g(G) \in \Omega(\log n)$ and $\text{tw}(G) \in \Omega(n)$ (using for example the results in [14] and [4]). Choosing k such that $k \cdot g(G) \in [n + 1, 2n]$ one obtains a graph which does not contain k vertex-disjoint cycles (of any length) but whose tree-width is in $\Omega(k \log k)$.

2 Proof of the main result

We will use the following lemma.

Lemma 2.1 (Diestel [7]). *For each natural number k , let*

$$s_k := \begin{cases} 4k(\log_2 k + \log_2 \log_2 k + 4) & \text{if } k \geq 2, \\ 1 & \text{if } k = 1. \end{cases}$$

Then every 3-regular multigraph on at least s_k vertices contains a set of k vertex-disjoint cycles.

Fix $l \geq 3$ and a graph G . We say that a cycle in G is *long* if it has length at least l , and otherwise we say that it is *short*. By *disjoint*, we always mean *vertex-disjoint*.

We assume that G does not contain k disjoint long cycles and show that G contains a transversal of \mathcal{F}_l of size at most $f(l, k)$, where

$$f(l, k) = \begin{cases} 6kl + 10k \log_2 k + 40k + 10k \log_2 \log_2 k & \text{if } k \geq 2, \\ 0 & \text{if } k = 1. \end{cases}$$

The proof is by induction on k , where the base case $k = 1$ is obvious.

If G contains a long cycle C of length at most $6l$ then by induction, $G - V(C)$ contains either $k - 1$ disjoint long cycles or a transversal X of size $f(l, k - 1)$. In the first case G contains k disjoint long cycles and in the second case $X \cup V(C)$ is a transversal of size $f(l, k - 1) + 6l \leq f(l, k)$. Therefore we may assume that every long cycle in G contains strictly more than $6l$ vertices.

Let H denote a maximal subgraph of G with the following properties:

1. all vertices of H have degree 2 or 3 in H ;
2. H contains no short cycle.

Similarly as in [9], observe that H is the union of a subdivision of a 3-regular multigraph and at most $k - 1$ disjoint long cycles. If H contains at least s_k vertices of degree 3 then by Lemma 2.1, it contains k disjoint cycles, which by definition of H are all long. So from now on, we can assume that H contains fewer than s_k vertices of degree 3.

Definition 2.2. *We say that a path P is an H -path if its endpoints are distinct vertices of H and if it is internally vertex-disjoint from H . Observe that we allow for $P \subseteq H$ if the length of P is one. We say that P is a proper H -path if none of its edges are contained in H .*

For each $i \in \{2, 3\}$ let $V_i \subseteq V(H)$ denote the set of vertices with degree i in H . We modify G by removing all edges from $E(G) \setminus E(H)$ that are incident to a vertex from V_3 . Note that any transversal of the modified graph can be turned into a transversal of the original graph by additionally removing V_3 . Furthermore H is still maximal in the modified graph. From now on we assume that all vertices of V_3 have degree 3 in G . In particular the endpoints of every proper H -path lie in V_2 .

This implies that every H -path has length at most l , as otherwise we could add the path to H without violating either the degree or the cycle condition, contradicting the maximality. For the same reason, if P is an H -path with endpoints $s, t \in V(H)$, then there exists a path between s and t in H of length at most l . In fact, as H contains no cycles of length at most $2l$, this path is unique. Thus the following notion is well-defined.

Definition 2.3 (Projection). *Suppose that P is an H -path with endpoints $s, t \in V(H)$. The projection of P , denoted by $\pi(P)$, is defined to be the unique path of length at most l between s and t in H .*

Let $C \subseteq G$ be a cycle in G that intersects H in at least two vertices. We define the projection $\pi(C)$ of C as follows. Let $C = P_1 \cup \dots \cup P_m$ be a decomposition of C into distinct H -paths. Then we define the projection of C to be the multigraph $\pi(C) = \pi(P_1) \cup \dots \cup \pi(P_m)$.

If P is a path in G with distinct endpoints in H (not necessarily an H -path), then we define the projection analogously: let $P = P_1 \cup \dots \cup P_m$ be a decomposition into H -paths and define $\pi(P) = \pi(P_1) \cup \dots \cup \pi(P_m)$.

We remark that in the definition above, the decomposition of a cycle or path into distinct H -paths is unique up to permutation, so that the projection is in fact well-defined.

We claim that information about the length of a cycle can be recovered by looking at the following property of its projection:

Definition 2.4. *A multigraph M is called even if the multiplicity of every edge in M is even.*

Lemma 2.5. *Suppose that C is a cycle in G which intersects H in at least two vertices. If $V(\pi(C))$ induces a tree in H , then C is short. If $\pi(C)$ is not even, then C is long.*

Proof. Among all cycles for which the lemma fails, we may pick a cycle C whose decomposition into H -paths minimizes the number of proper H -paths. If none of these H -paths is proper, then $C \subseteq H$ and thus $\pi(C) = C$ is not even and long. Therefore we may assume that C contains at least one proper H -path P .

Note that $\pi(P)$ is a path in H with the same endpoints as P . If C contains all edges of $\pi(P)$, then we actually have $C = P \cup \pi(P)$ and so the projection of C is even. Moreover, the length of C is $|P| + |\pi(P)| \leq 2l$. Since all long cycles have length greater than $6l$ we see that C must be short and we are done. Therefore, we can assume that at least one edge of $\pi(P)$ does not belong to C . Since $\pi(P)$ is a path with endpoints in $V(C)$, there exists a path $P' \subseteq \pi(P)$ with endpoints $s, t \in C$ which is internally vertex-disjoint from C and whose edges are not edges of C . Let $P_1, P_2 \subseteq C$ denote the two internally disjoint s, t -paths in C and consider the two cycles $C_1 := P_1 \cup P'$ and $C_2 := P_2 \cup P'$. Observe that since $\pi(P') \subseteq \pi(C)$ we have $V(\pi(C)) = V(\pi(C_1)) \cup V(\pi(C_2))$. Additionally, the parity of each edge in $\pi(C)$ is equal to the parity of the same edge in $\pi(C_1) \cup \pi(C_2)$.

We are now ready to prove the first claim. Assume that C is long and that $H[V(\pi(C))]$ is a tree. Then the projections of C_1 and C_2 induce trees as well and therefore both cycles contain at least one proper H -path. In particular both cycles contain strictly fewer proper H -paths than C . Furthermore, at least one of the two cycles has length at least $|C|/2$. Since C has length at least $6l$, this cycle is still long, which contradicts the minimality of C .

For the second claim, assume that C is short and that $\pi(C)$ is not even. Observe that we have $|C_i| \leq |C| + |P'| \leq l + |P'|$ for each $i \in \{1, 2\}$. As P' is a subgraph of $\pi(P)$ we know that $|C_i| \leq 2l$, which implies that both cycles C_i are short. Since H contains only long cycles, this means that both C_1 and C_2 contain at least one proper H -path, so both C_1 and C_2 contain fewer proper H -paths than C . Finally, as $\pi(C)$ is not even and since each edge in $\pi(C_1) \cup \pi(C_2)$ has the same parity as the same edge in $\pi(C)$, at least one of $\pi(C_1)$ and $\pi(C_2)$ is not even. This contradicts the minimality of C . \square

Definition 2.6. *Let $X \subseteq V(H) \cup E(H)$ be a set of vertices and edges of H . Let us denote by $G - X$ and $H - X$ the graphs obtained by deleting from G and H the edges and vertices in X .*

Then we say that X is π -preserving if $P \subseteq G - X$ implies $\pi(P) \subseteq H - X$ for all H -paths P .

The following lemma is the crucial ingredient of the proof of Theorem 1.1. For now, we only state the lemma; the proof is given in the next subsection.

Lemma 2.7. *There exists a π -preserving set X for which $H - X$ is a forest and such that $|X \cap (V_2 \cup E(H[V_2]))| \leq 3|V_3|/2 + k$.*

We can now finish the proof of Theorem 1.1. By the lemma, there exists a π -preserving set X for which $H - X$ is a forest and such that $|X \cap (V_2 \cup E(H[V_2]))| \leq 3|V_3|/2 + k$. Suppose that C is a cycle that intersects H at least twice. Then because X is π -preserving, we know in particular that all vertices of $V(\pi(C))$ are contained in the same component of $H - X$. Since each component of $H - X$ is a tree, the graph $H[V(\pi(C))]$ must also be a tree. Thus, by Lemma 2.5, the cycle C is short. It follows that every long cycle in $G - X$ intersects H at most once.

To construct the transversal we define the following two sets.

1. Let $X' \subseteq V(H)$ be a set containing the vertices in $X \cap V_2$ and also containing one endpoint of each edge in $X \cap E(H[V_2])$.
2. Let $Z \subseteq V(H)$ denote the set of all vertices $z \in V(H)$ for which there exists some long cycle C_z such that $V(C_z) \cap V(H) = \{z\}$.

We now claim that $V_3 \cup X' \cup Z$ is a transversal of all long cycles. Every long cycle C intersects H at least once since otherwise C could be added to H . If C intersects H exactly once then it intersects Z . If C intersects H at least twice then, by the observation above, this means that C intersects X . But then C must intersect either V_3 or X' . So $V_3 \cup X' \cup Z$ is a transversal in G . Recall that in the beginning, we modified the graph G by removing all edges of $E(G) \setminus E(H)$ that are incident to a vertex of V_3 . Since we remove V_3 anyway, $V_3 \cup X' \cup Z$ is also a transversal in the original graph.

It remains to bound the size of this transversal. If for some $z \neq z' \in Z$ the cycles $C_z, C_{z'}$ intersect, then one can see that the assumption $|C_z|, |C_{z'}| \geq 6l$ implies that $C_z \cup C_{z'}$ contains a z - z' -path of length at least l , which contradicts the fact that there are no H -paths of length l or longer. Therefore $\{C_z \mid z \in Z\}$ is a collection of vertex disjoint long cycles and in particular $|Z| < k$. Furthermore, we have $|X'| \leq 3|V_3|/2 + k$. Since $|V_3| < s_k$, we get

$$|V_3 \cup X' \cup Z| < |V_3| + \frac{3}{2}|V_3| + 2k < \frac{5}{2}s_k + 2k \leq f(l, k).$$

This completes the proof of Theorem 1.1. However, we still need to prove Lemma 2.7.

2.1 Proof of Lemma 2.7

Let us call a π -preserving set *valid* if it contains at most one edge or vertex from every component of $H[V_2]$. There exists at least one valid π -preserving set: the empty set. Let X denote a π -preserving set of maximal size. Since H is the disjoint union of a subdivision of a 3-regular graph on $|V_3|$ vertices and fewer than k cycles, the fact that X is valid immediately implies that $|X \cap (V_2 \cup E(H[V_2]))| \leq 3|V_3|/2 + k$. We will show that $H - X$ is a forest. Assume towards a contradiction that $H - X$ contains a cycle. Among all such cycles let $C = (x_0 \dots, x_n)$ denote one of minimum length. Since H only contains long cycles, we have $|C| > 6l$. Let $B_{H-X}(C, l)$ denote the ball of radius l around C in $H - X$, i.e., the set of vertices that have distance at most l to a vertex of C in $H - X$. Let H_C be the subgraph of $H - X$ induced by $B_{H-X}(C, l)$. We now have some claims about the structure of H_C . First of all, it is clear that $C \subseteq H_C$. Moreover:

Claim 2.8. *In H_C , every vertex v has a unique nearest vertex in C (which may be v itself if $v \in V(C)$).*

Proof. This is clear if $v \in V(C)$, so assume otherwise. By the definition of H_C , the distance from v to any nearest vertex on C is at most l . If there are two nearest vertices of v on C , then using the shortest path between them on C , we obtain a cycle of length at most $|C|/2 + 2l < |C|$ in H_C , where the inequality follows from $|C| > 4l$. But this would contradict the minimality of C . \square

This claim shows in particular that the projection map $p: V(H_C) \rightarrow V(C)$ taking vertices of H_C to their unique nearest vertex in C is well-defined. In fact, the preimages of this projection have rather nice properties:

Claim 2.9. *The following hold:*

- (i) *for every $x \in V(C)$, the graph $H_C[p^{-1}(x)]$ is a tree;*
- (ii) *for distinct vertices $x, y \in C$, there are no edges between $p^{-1}(x)$ and $p^{-1}(y)$ in $H_C \setminus C$;*

Proof. For (i) simply observe that the diameter of $H_C[p^{-1}(x)]$ is at most $2l$, so any cycle would be of length at most $4l$; however H does not contain cycles that are this short. The argument for (ii) is similar to the argument in the proof of Claim 2.8: if such an edge exists, then H_C contains a cycle of length at most $|C|/2 + 2l + 1 < |C|$, using $|C| > 6l > 4l + 2$. This would contradict the minimality of C . \square

By the above claim, the graph H_C is just the cycle C with trees attached at every vertex. In particular, $H_C - e$ is a tree for any edge $e \in E(C)$.

We will now construct a larger graph G_C where $H_C \subseteq G_C \subseteq G - X$ as follows. Let $D = \{x_l, x_{l+1}, \dots, x_{n-l}\}$. For each $x \in D$, let $P(x)$ be the set of all H -paths P in $G - X$ such that $x \in V(\pi(P))$. Since X is π -preserving, every such path satisfies $\pi(P) \subseteq H - X$. We then define

$$G_C = H_C \cup \bigcup_{x \in D} P(x).$$

It is worth noting that $G_C \cap H = H_C$. Let us further denote by e^* the edge $\{x_0, x_n\}$. We have the following very important claim, whose proof we postpone to the end of the section.

Claim 2.10. *For every proper H -path $P \subseteq G_C$, we have $\pi(P) \subseteq H_C - e^*$. In particular, every path in $G_C - e^*$ with endpoints in $V(H_C)$ projects to a subgraph of $H_C - e^*$.*

Using the claim, we can now finish the proof of Lemma 2.7. Let $A = \{x_0, \dots, x_l\}$ and $B = \{x_{n-l}, \dots, x_n\}$. We claim that $G_C - e^*$ does not contain two internally disjoint A - B -paths. Suppose for a contradiction that P_1 and P_2 are two such paths, where we can assume that both P_1 and P_2 intersect both A and B in exactly one vertex. By Claim 2.10 both paths project onto a subgraph of $H_C - e^*$, which implies in particular that $\pi(P_1) \cup \pi(P_2)$ does not contain e^* . We can combine P_1 and P_2 into a cycle in $G_C - e^*$ by adding the shortest paths between the endpoints of P_1 and P_2 in $C[A]$ and $C[B]$, respectively. The projection of this cycle lies in $H_C - e^*$ which is a tree. Thus, by Lemma 2.5, this cycle is short. In particular we have $|P_1| < l$. However, as the

projection of P_1 avoids e^* , connecting the endpoints of P_1 using the shortest path in $C[A \cup B]$ results in a cycle C_1 whose projection is not even, since the multiplicity of e^* in $\pi(C_1)$ is one. So by Lemma 2.5 the cycle C_1 is long. Since any path in $C[A \cup B]$ is of length at most $2l + 1$, the length of C_1 is bounded by $|P_1| + 2l + 1 \leq 3l$. This is a contradiction since G does not contain a long cycle of length at most $3l$. We conclude that there are no two internally disjoint A - B -paths in $G_C - e^*$.

By Menger's theorem, $G_C - e^*$ contains a single-vertex A - B -cut. Denote the cut vertex by x . Because C contains an A - B -path from x_l to x_{n-l} , we must have $x \in D$. Thus x has degree at least two in $H_C - e^*$. We distinguish two cases, depending on whether x has degree two or three in $H_C - e^*$.

First, assume that x is a vertex of degree two in $H_C - e^*$. By the maximality of X , we know that $X \cup \{x\}$ is not a valid π -preserving set. One possibility is that $X \cup \{x\}$ is π -preserving, but it is not valid. Since X is valid on its own, this means that x belongs to a component of $H[V_2]$ which intersects X . But since x belongs to the cycle $C \subseteq H - X$, this is impossible. Thus it must be that $X \cup \{x\}$ is not a π -preserving set. Since X by itself is π -preserving, the definition implies that there exists an H -path P in $G - X$ such that $x \in V(\pi(P))$ and $x \notin V(P)$. Note that $P \subseteq G_C - e^*$ by the definition of G_C . Since x has degree two in $H_C - e^*$ and since $H_C - e^*$ is a tree, we know that $H_C - e^* - x$ breaks into exactly two trees T_1 and T_2 . As $x \in V(\pi(P))$ and $\pi(P) \subseteq H_C - e^*$, it must be that P has one endpoint in T_1 and the other in T_2 . It follows that $T_1 \cup T_2 \cup P$ is a connected graph and thus $T_1 \cup T_2 \cup P$ must contain an A - B -path. This is a contradiction with the fact that x separates A and B in $G_C - e^*$ and completes the proof in this case.

Next, assume that x is a vertex of degree three in $H_C - e^*$. Recall that then x has degree 3 in G as well, and in particular no proper H -path has x as an endpoint. As in the previous case, $H_C - e^* - x$ breaks into exactly three trees T_1, T_2 and T_3 . Let x^- and x^+ be the neighbours of x on C . Without loss of generality assume that $x^- \in T_1$ and $x^+ \in T_3$. Consider any H -path $P \subseteq G - X$ such that $x \in V(\pi(P))$ (so in particular $P \subseteq G_C - e^*$). Since $\pi(P)$ is a path contained in $H_C - e^*$, the endpoints of P must be in two different trees. However, it cannot happen that one endpoint of P is in T_1 and the other in T_3 , since then $T_1 \cup T_3 \cup P$ would contain an A - B -path, contradicting the fact that x is an A - B -cut vertex in $G_C - e^*$. For the same reason there are no two H -paths $P_1, P_2 \subseteq G - X$ such that $x \in V(\pi(P_1)) \cap V(\pi(P_2))$ and such that P_1 has endpoints in T_1 and T_2 and P_2 has endpoints in T_2 and T_3 . We conclude that there is some $j \in \{1, 3\}$ such that all H -paths P with $x \in V(\pi(P))$ have one endpoint in T_2 and the other endpoint in T_j . Without loss of generality, assume $j = 1$. We now claim that by adding the edge $\{x, x^+\}$ to X we get again a π -preserving set. If we assume otherwise, then there exists a H -path P in $G - (X \cup \{\{x, x^+\}\})$ whose projection contains $\{x, x^+\}$. This is not possible, as such a path satisfies $x \in V(\pi(P))$ and has one endpoint lying in T_3 . Thus $X \cup \{\{x, x^+\}\}$ is π -preserving. In fact, since the edge $\{x, x^+\}$ is incident to the degree-three vertex x , it is automatically a valid π -preserving set. This contradicts the maximality of X and completes the proof of Lemma 2.7.

We now present the missing proof of Claim 2.10.

Proof of Claim 2.10. The second statement clearly follows from the first.

We start the proof with an observation about the H -paths in $G - X$ whose endpoints lie in $V(H_C)$. Let $P \subseteq G - X$ be such a path with endpoints $a, b \in V(H_C)$. Since

X is π -preserving, we have $\pi(P) \subseteq H - X$. Note that we know of at least one path in $H - X$ from a to b : the path $Q = Q_1 Q_2 Q_3$, where Q_1 is the shortest path in H_C from a to $p(a)$, Q_2 is the shortest path from $p(a)$ to $p(b)$ on C , and Q_3 is the shortest path from $p(b)$ to b in H_C . This path Q is contained in H_C and it has length at most $2l + |C|/2$. The important observation is that $\pi(P) = Q$. Indeed, if this were not so, then $Q \cup \pi(P) \subseteq H - X$ would contain a cycle of length at most $2l + |C|/2 + |\pi(P)| \leq 3l + |C|/2 < |C|$ (using $|C| > 6l$), contradicting the minimality of C .

This shows in particular that for every H -path $P \subseteq G - X$ with endpoints in $V(H_C)$, we have $\pi(P) \subseteq H_C$. It remains to show the stronger statement that if additionally $P \subseteq G_C$, then $\pi(P) \subseteq H_C - e^*$.

To obtain a contradiction, assume that $P \subseteq G_C$ is an H -path with endpoints $a, b \in V(H_C)$ such that $\{x_0, x_n\} \in E(\pi(P))$. Since the projection of P has length at most l , and by the observation above, we can assume without loss of generality that $p(a) \in \{x_0, \dots, x_l\}$ and $p(b) \in \{x_{n-l}, \dots, x_n\}$. Now let c be the neighbour of a on P . The edge $\{a, c\}$ belongs to G_C but not to H_C , so there is some $x \in D = \{x_l, \dots, x_{n-l}\}$ and some H -path $P' \in P(x)$ such that $\{a, c\} \in E(P')$. Let d be the other endpoint of P' (one endpoint is a). Since $|C| > 6l$, $x \in D$, $x \in V(\pi(P'))$, and $p(a) \in \{x_0, \dots, x_l\}$, we must have $p(d) \in \{x_l, \dots, x_{2l}\}$. The union of P and P' contains an H -path with endpoints d and b . However, the projection of this H -path has length at least $\min\{l+1, |C| - 3l\} > l$, which is impossible. \square

3 Conclusions

The main contribution of the paper is showing the asymptotically optimal Erdős-Pósa function for the case of long cycles and growing k and ℓ and thus answering the question asked in [3] and [11]. We conclude the paper by mentioning some open problems:

- **S -cycles:** Kakimura, Kawarabayashi, and Marx [12] introduced a different generalization of the standard Erdős-Pósa theorem. They considered the family of S -cycles, i.e., all cycles of a graph which intersect a specified set S , and proved that such a family of cycles has the Erdős-Pósa property. Their result was later improved by Pontecorvi and Wollan [16], resulting in the following theorem:

Theorem 3.1. *For any graph and any vertex subset S , the graph either contains k vertex-disjoint S -cycles or a vertex set of size $\mathcal{O}(k \log k)$ that meets all S -cycles.*

Since the vertex set S can be the vertex set of the whole graph, this result is asymptotically tight. In 2014, Bruhn, Joos, and Schaudt [5] combined the family of S -cycles with the family of long cycles and proved that the family of S -cycles of length at least ℓ has the Erdős-Pósa property with $f(k, \ell) = \mathcal{O}(\ell k \log k)$. Thus, it is natural to ask if Theorem 1.1 generalizes to S -cycles as well.

- **Edge-version:** A family of graphs \mathcal{F} is said to have the *edge-Erdős-Pósa* property if there exists a function $f: \mathbb{N} \rightarrow \mathbb{R}$ such that every graph G which does not contain k edge-disjoint members of \mathcal{F} contains a set of $f(k)$ edges which meets all copies of members of \mathcal{F} in G . Since the pioneering papers by Erdős and Pósa [8, 9] it has been known that the family of cycles has the edge-Erdős-Pósa property as well. Namely, the following is true:

Theorem. *Any graph G contains either k edge-disjoint cycles or a set of $(2 + o(1))k \log k$ edges meeting all its cycles.*

Pontecorvi and Wollan [16] generalized this result to the case of S -cycles by a clever reduction to the standard vertex-version of the problem. Already Birmelé et al. [3] asked if the family of long cycles has the edge-Erdős-Pósa property. Unfortunately, the gadget trick from [16] breaks down in the case of long cycles and thus nothing is known in this scenario. It would be interesting to see if our approach could be applied for proving that the family of long cycles has the edge-Erdős-Pósa property.

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